E. Other expressions followed from $|a_2(t)|^2$

(a) Allowing for a spread in frequency in the incident light

$\frac{1}{2}\varepsilon_0 \mathcal{E}_0^2$ (for incident frequency ω) = energy density $\equiv U_\omega$

["ε_0" in "$4\pi\varepsilon_0$" in EM]

[unit: energy per unit volume]

$$\therefore\ |a_2(t)|^2_\omega = \frac{2}{\varepsilon_0} U_\omega e^2 |z_{21}|^2 \frac{\sin^2\left[\frac{(E_2 - E_1 - \hbar\omega)t}{2\hbar}\right]}{(E_2 - E_1 - \hbar\omega)^2} \qquad (26)$$

(From Eq. (23))

- Incident light with a spread in ω's:

$U(\omega)\,d\omega$ = energy density in the interval of frequencies from ω to $\omega + d\omega$

<u>Idea:</u> Treat each frequency independently $\Rightarrow \begin{cases} |a_2(t)|^2_\omega \\ \vdots \\ |a_2(t)|^2_{\omega'} \end{cases}$

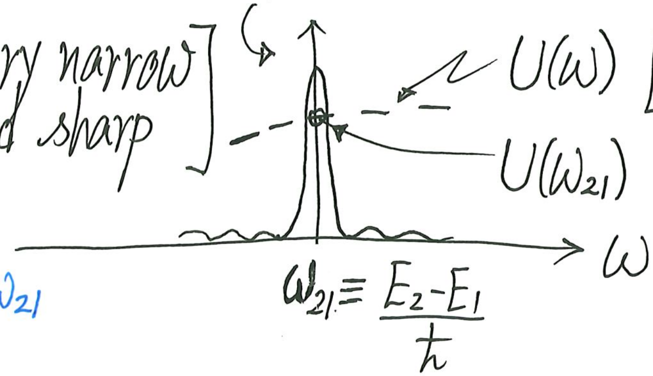
then <u>add</u> up $|a_2(t)|^2$ for different ω's

This works for <u>non-monochromatic</u> <u>incoherent</u> EM waves OR "<u>incoherent</u> <u>perturbations</u>"

$$|a_2(t)|^2_{\omega\to\omega+d\omega} = \frac{2}{\varepsilon_0} e^2 |z_{21}|^2 \underbrace{\overbrace{\frac{\sin^2\left[\frac{(E_2-E_1-\hbar\omega)}{2\hbar} t\right]}{(E_2-E_1-\hbar\omega)^2}}^{\text{like a delta function}}} U(\omega)\,d\omega \quad (27)$$

area under curve $\sim t$

$$\frac{\pi}{2\hbar^2} t\, \underbrace{\delta\left(\frac{E_2-E_1}{\hbar} - \omega\right)}_{\text{sharp peak at } \omega=\omega_{21}} \leftarrow \left[\begin{array}{l}\text{very narrow} \\ \text{and sharp}\end{array}\right]$$



Adding up contributions from all ω's in incident light gives

$$|a_2(t)|^2 = \frac{2e^2}{\varepsilon_0}|z_{21}|^2 \underbrace{\int_0^\infty \frac{\sin^2\left[\frac{(E_2-E_1-\hbar\omega)}{2\hbar}t\right]}{(E_2-E_1-\hbar\omega)^2}}_{\text{Non-zero contribution to integral}^\dagger \text{ only at } \omega=\omega_{21}\equiv\frac{E_2-E_1}{\hbar}} U(\omega)\,d\omega$$

"the right ω"

$$= \frac{2e^2}{\varepsilon_0}|z_{21}|^2 \frac{\pi}{2\hbar^2} t\, U(\omega_{21})$$

[integrate $U(\omega)$ with $\delta(\omega_{21}-\omega)$ over ω]

$$= \frac{\pi}{\varepsilon_0\hbar^2} U(\omega_{21})\, e^2 |z_{21}|^2 \cdot t \qquad (28)$$

picks up strength of light at the right ω_{21}

selection rule

linear in t

Now, $\dfrac{|a_2(t)|^2}{t}$ is a quantity of unit $\left(\dfrac{1}{\text{time}}\right)$

- The Rate at which transition $(1\to 2)$ occurs

$= $ Transition Probability per unit time

$$\equiv \lambda_{1\to 2} = \frac{|a_2(t)|^2}{t} = \frac{\pi e^2}{\varepsilon_0 \hbar^2} U(\omega_{21}) |z_{21}|^2 \quad (29)$$

(non-monochromatic) (for $\hat{z}$-polarized light)

lower $\to 1$, higher $\to 2$

averaging over polarizations and propagation directions[†]

Generally, Transition rate

$$\boxed{\lambda_{1\to 2} = \frac{\pi e^2}{3\varepsilon_0 \hbar^2} U(\omega_{21}) |r_{21}|^2} \quad (30)$$

- $|r_{21}|^2 \equiv |\vec{r}_{21}|^2$ with $\vec{r}_{21} = \int \psi_2^*(\vec{r})\, \vec{r}\, \psi_1(\vec{r})\, d^3r$, $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$

- $e^2 |\vec{r}_{21}|^2 \equiv |\vec{\mu}_{21}|^2$ (reminds us that it is electric dipole moment that matters)

[†] Don't worry about the details from (28) to (29). They carry the same physics.

If "1" is the higher state ("2" before) and "2" is the lower state ("1" before), it is the case of <u>stimulated emission</u>

Following same steps:

$$\underset{\text{higher}\quad\text{lower}}{\lambda_{1\to 2}} = \frac{\pi e^2}{3\varepsilon_0 \hbar^2} U(\omega_{12}) \, |r_{21}|^2 \qquad (31)$$

For the same states (upper & lower), $|r_{21}|^2$ in (30) and (31) are the same.

$\because$ Same 2 levels under same condition ($U(\omega_{21})$ the same),

$$\boxed{\begin{aligned}&\text{transition rate of stimulated absorption}\\ &\qquad = \text{transition rate of stimulated emission}\end{aligned}} \qquad (32)$$

(Here, this result is obtained by QM)

(b) Allowing for a group of "almost degenerate states" as "2" (Optional) [Try it yourself]

"2" - - - - - - - - - - - - OR (many states as "2") (e.g. in solids) [degenerate OR almost degenerate]

"1" —— initially $a_1(0) = 1$

$$\underbrace{D(E_2)}_{\text{called "density of states"}} dE_2 = \# \text{ states with energy in interval } E_2 \text{ to } E_2 + dE_2$$

How to develop an expression for $|a_2(t)|^2$ and transition rate, given monochromatic incident at ω?

Physical Sense: derived $|a_2(t)|^2$ for one state "2". Many state "2"? Add them up?

Remarks

- Eqs. (29), (30), (31) are special forms of Fermi Golden Rule
- λ has units of $\frac{1}{\text{time}}$ (s^{-1})
- $\frac{\lambda}{U(\omega_{12})}$ has units of $\frac{1}{(\text{time})} \cdot \frac{1}{\left(\frac{\text{energy}}{\text{Volume}}\right)\frac{1}{(\text{freq.})}} = \frac{1}{(\text{time})^2 \cdot \left(\frac{\text{energy}}{\text{Volume}}\right)}$

$$\left[U(\omega)d\omega = \text{energy density} = \frac{\text{energy}}{\text{Volume}} \right]$$

- From Fermi Golden Rule (or $\lambda_{1\to2}$, etc.), one can calculate measurable quantities such as spectroscopic absorption coefficient, life time of an atomic state, cross section, etc.

Refs:

- For time-dependent perturbation theory
  - See Ch. 9 of Griffiths (we added in more physical descriptions)
  - See Ch. 11 of Yariv's "An introduction to Theory and Applications of Quantum Mechanics"